MATH 2028 Surface Integrals in $\mathbb{R}^3$

GOAL: Define surface integrals of functions and vector fields on surfaces in $\mathbb{R}^3$

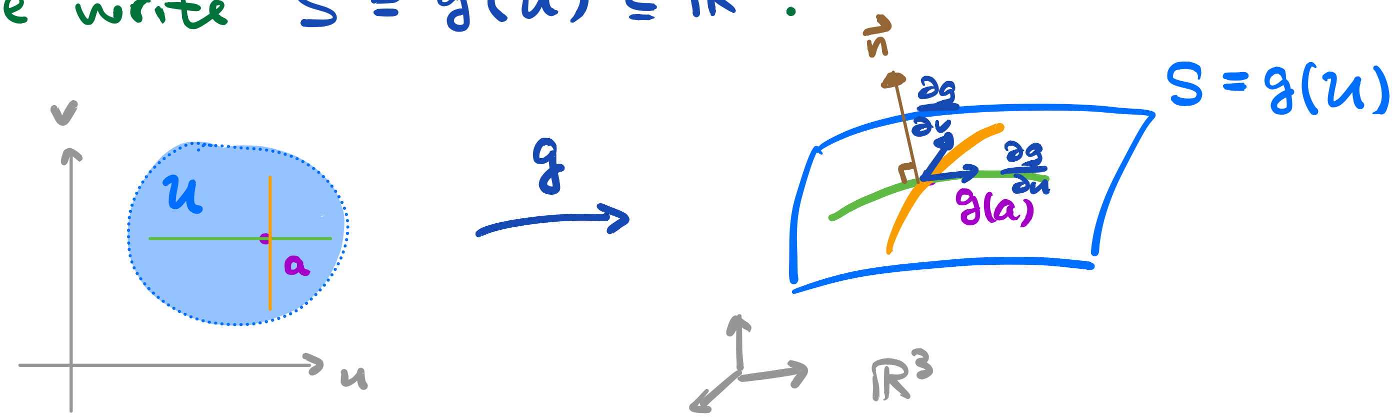
Q: What is a "surface"?

Def$^{\underline{n}}$: A regular parametrized surface in $\mathbb{R}^3$ is a map $g: \mathcal{U} \to \mathbb{R}^3$ where $\mathcal{U} \subseteq \mathbb{R}^2$ is a bounded open subset s.t.

(i) g is an injective C^1 map

(ii) $Dg(a): \mathbb{R}^2 \to \mathbb{R}^3$ has rank 2 $\forall a \in \mathcal{U}$

We write $S = g(\mathcal{U}) \subseteq \mathbb{R}^3$.



Remark: Write $g = g(u,v)$

$$Dg = \begin{pmatrix} | & | \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ | & | \end{pmatrix} \text{ has rank 2}$$

$\Longleftrightarrow \left\{ \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v} \right\}$ linearly independent in $\mathbb{R}^3$

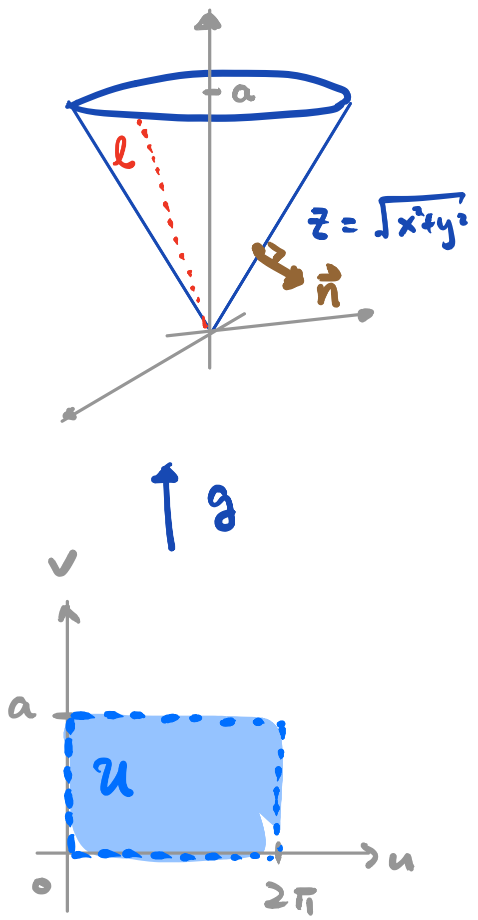
$$\Longleftrightarrow \quad \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \neq \vec{0}$$

We define the **unit normal of S** (w.r.t. $g(u,v)$)

as

$$\vec{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|}$$

Example 1 : (cone)



$$g : \underbrace{(0, 2\pi) \times (0, a)}_{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

$$g(u,v) = (v\cos u, v\sin u, v)$$

Claim: This is a regular parametrized surface with $S = \text{cone} \setminus \ell$.

- g is C^1 and 1-1
- $$\left.\begin{aligned} \frac{\partial g}{\partial u} &= (-v\sin u, v\cos u, 0) \\ \frac{\partial g}{\partial v} &= (\cos u, \sin u, 1) \end{aligned}\right\} \text{linearly indep.}$$

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = (v\cos u, v\sin u, -v)$$

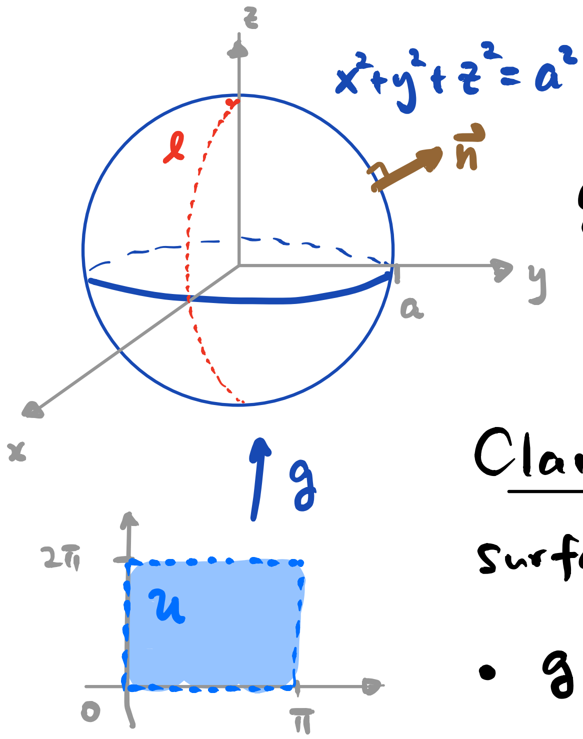
$$\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| = \sqrt{2}\, v > 0 \qquad (\because v \in (0,a))$$

$\therefore$ point downward

- $$\vec{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|} = \frac{1}{\sqrt{2}} (\cos u, \sin u, -1)$$

Example 2 : (sphere)

Fix $a > 0$.



$$g : \underbrace{(0,\pi)\times(0,2\pi)}_{\mathcal{U}\subseteq\mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

$$g(\phi,\theta) = (a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$$

"spherical coordinates"!

Claim: This is a regular parametrized surface with $S = \text{sphere} \setminus \ell$.

- g is C^1 and 1-1

$$\frac{\partial g}{\partial \phi} = (a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi)$$

$$\frac{\partial g}{\partial \theta} = (-a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0)$$

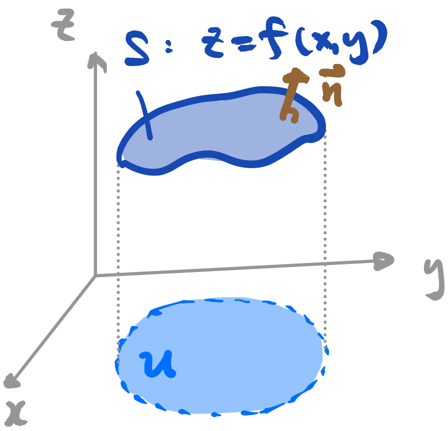
$$\frac{\partial g}{\partial \phi}\times\frac{\partial g}{\partial \theta} = (a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi)$$

$$\left\|\frac{\partial g}{\partial \phi}\times\frac{\partial g}{\partial \theta}\right\| = a^2\sin\phi > 0 \quad (\because\ \phi\in(0,\pi))$$

$$\vec{n} = \frac{\frac{\partial g}{\partial \phi}\times\frac{\partial g}{\partial \theta}}{\left\|\frac{\partial g}{\partial \phi}\times\frac{\partial g}{\partial \theta}\right\|} = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

points "out" of the sphere

Example 3 : (Graphical surface)



Given a C^1 fcn $f : \mathcal{U} \to \mathbb{R}$, we have the graph of f given by the surface S parametrized by

$$g(u,v) : \mathcal{U} \longrightarrow \mathbb{R}^3$$

$$g(u,v) = (u, v, f(u,v))$$

Note that g is 1-1 and C^1.

$$\left.\begin{aligned} \frac{\partial g}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\ \frac{\partial g}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right) \end{aligned}\right\}$$

always linearly independent!

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

$$\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2} = \sqrt{1 + |\nabla f|^2} > 0$$

$$\vec{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|} = \frac{1}{\sqrt{1 + |\nabla f|^2}} \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right)$$

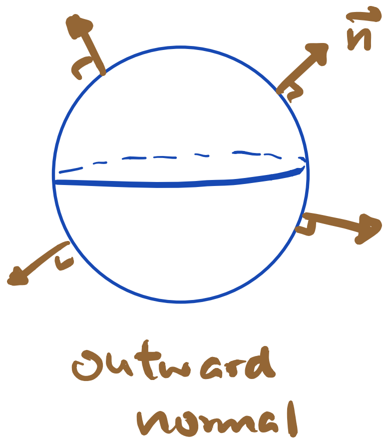
points "upward"

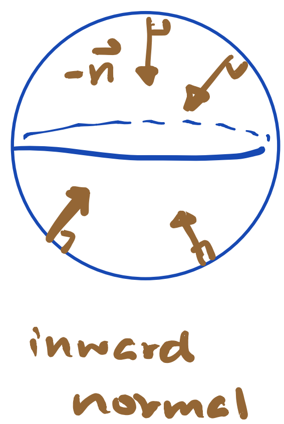
Q: How to define "orientation" of a surface?

Def^{n}: An orientation on $S \subseteq \mathbb{R}^3$ is a globally defined unit normal $\vec{n} : S \to \mathbb{R}^3$ which is continuous on all of S.

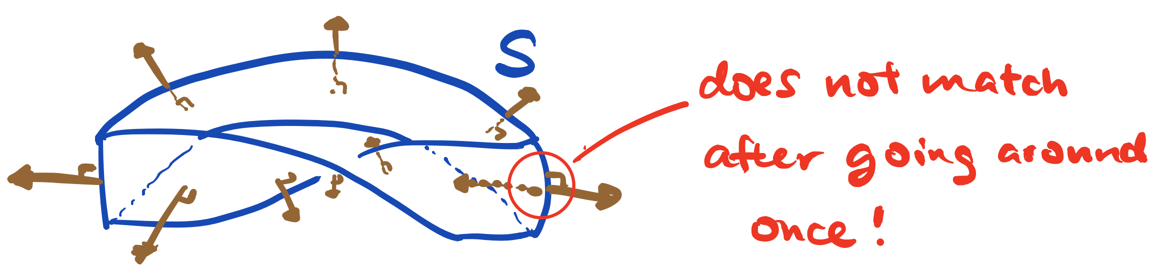
If such an $\vec{n}$ exists, we say that S is orientable. Otherwise, it is called non-orientable.

Example: The sphere is orientable with two possible orientations:





Example: A famous example of non-orientable surface is the Möbius strip



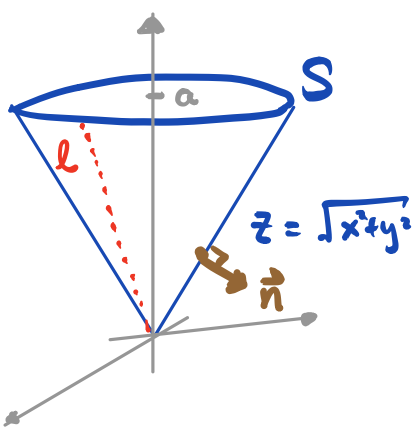
Def$^{\underline{n}}$: Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $g(u,v) : \mathcal{U} \to \mathbb{R}^3$ and $f : S \to \mathbb{R}$ be a cts function. THEN: the surface integral of f over S is

$$\int_S f \, d\sigma := \iint_{\mathcal{U}} (f \circ g) \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dA$$

In particular, the surface area of S is

$$\text{Area}(S) := \int_S 1 \, d\sigma$$

Example 1 : (cone)



$$g : \underbrace{(0,2\pi) \times (0,a)}_{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

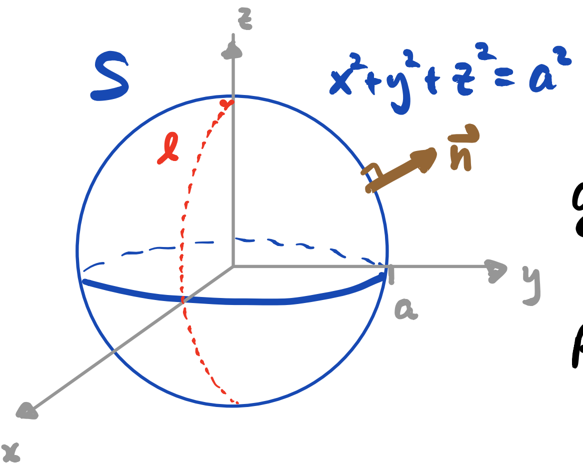
$$g(u,v) = (v\cos u, v \sin u, v)$$

$$\text{Area}(S) = \iint_{\mathcal{U}} \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| dA$$

$$= \int_0^{2\pi} \int_0^a \sqrt{2}\, v \, dv \, du = \sqrt{2}\, \pi a^2$$

$\square$

Example 2 : (Sphere)



$$g : \underbrace{(0,\pi)\times(0,2\pi)}_{\mathcal{U}\subseteq\mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

$$g(\phi,\theta) = (a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$$

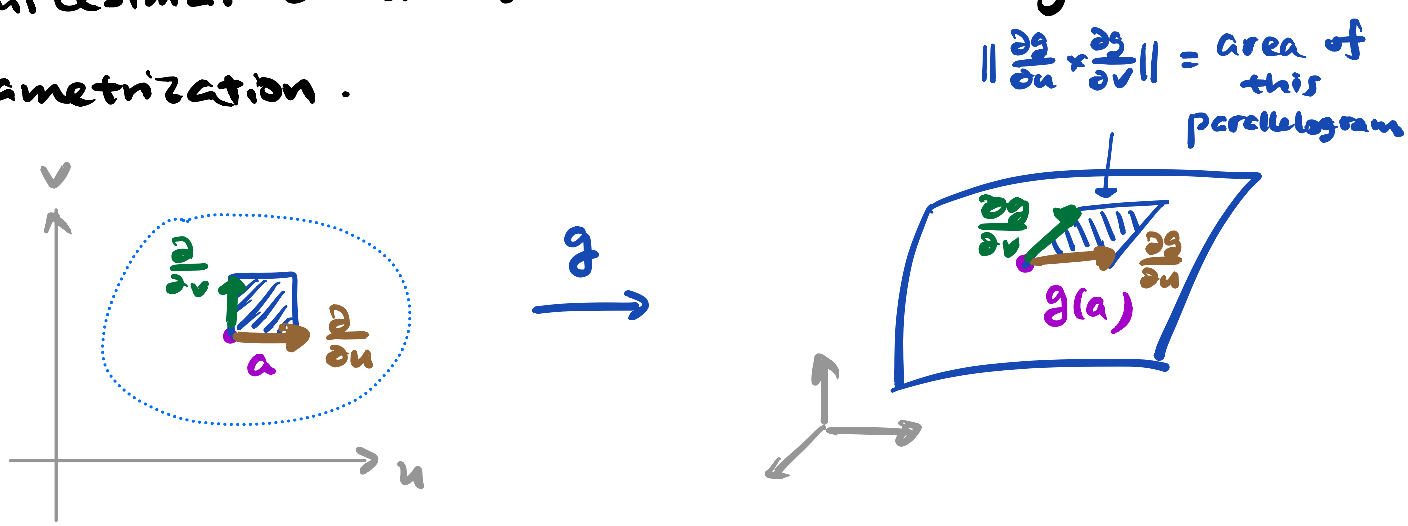
$$\text{Area}(S) = \iint_{\mathcal{U}} \left\| \frac{\partial g}{\partial \phi} \times \frac{\partial g}{\partial \theta} \right\| dA$$

$$= \int_0^{\pi}\int_0^{2\pi} a^2 \sin\phi \, d\theta \, d\phi = 4\pi a^2$$

$\blacksquare$

Remark: The surface integral $\int_S f\, d\sigma$ depends only on the surface $S = g(\mathcal{U})$ BUT not on the actual parametrization g.

Why? The $\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|$ term describes the infinitesimal "area distortion" caused by the parametrization.



Remark: (switch orientation of a surface)

Given a parametrized surface $S = g(\mathcal{U})$ given by $g(u,v) : \mathcal{U} \longrightarrow \mathbb{R}^3$, we can define another parametrized surface

$$\tilde{g}(v,u) : \tilde{\mathcal{U}} \to \mathbb{R}^3$$

by $\quad \tilde{g}(v,u) := g(u,v)$

where $\quad \tilde{\mathcal{U}} := \{ (v,u) \mid (u,v) \in \mathcal{U} \}$.

Def$^{\underline{n}}$: Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $g(u,v) : \mathcal{U} \to \mathbb{R}^3$ and $F : S \to \mathbb{R}^3$ be a cts vector field. THEN: the surface integral of F over S is

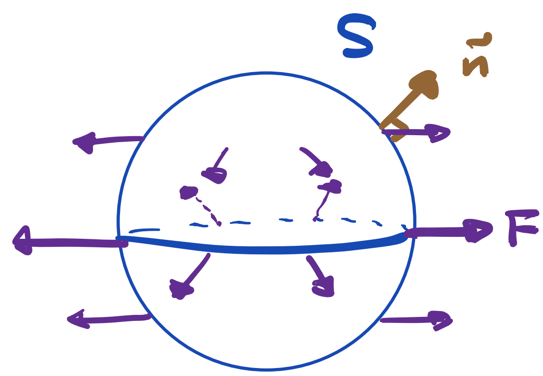
$$\int_S F \cdot \vec{n}\, d\sigma := \iint_{\mathcal{U}} (F \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) dA$$

We also call this the flux of F across S.

Remark: $\int_S F \cdot \vec{n}\, d\sigma$ is independent (up to a sign) of the parametrization of S.

Example 4: Compute the flux of the vector field $F(x,y,z) = (x,y,0)$ across the sphere S of radius $a>0$ centered at the origin, oriented by the outward normal.

Solution:



Step 1: Fix a parametrization.

$$g : \underbrace{(0,\pi) \times (0,2\pi)}_{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$

$$g(u,v) = (a\sin u \cos v,\ a \sin u \sin v,\ a\cos u)$$

points outward

Recall: $\dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v} = a^2 \sin u\,(\sin u \cos v,\ \sin u \sin v,\ \cos u)$

$$F \circ g = (a \sin u \cos v,\ a \sin u \sin v,\ 0)$$

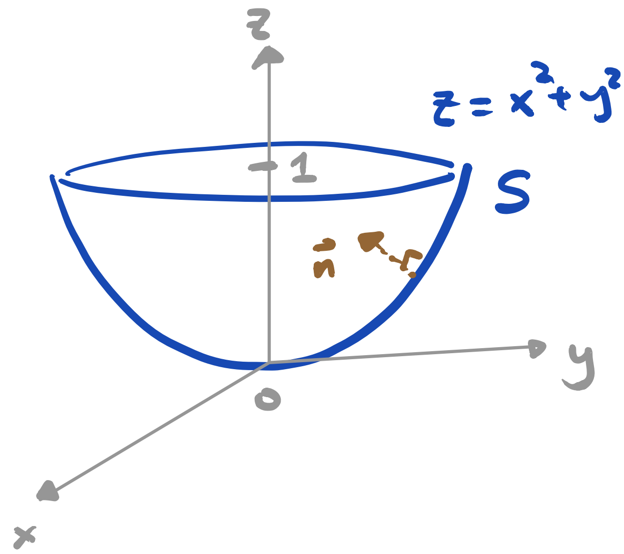
$$\Rightarrow (F \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) = a^3 \sin^3 u$$

Therefore,

$$\int_S F \cdot \vec{n}\, d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3 \sin^3 u\, du\, dv = \frac{8\pi}{3} a^3$$

$\square$

Example 5: Compute the flux of the vector field $F(x,y,z) = (y, -x, z^2)$ across the surface S which is the portion of paraboloid $z = x^2+y^2$ between $z=0$ and $z=1$, oriented by the "upward" pointing unit normal $\vec{n}$.



Solution: Note that S is the graph of of the function

$$f(x,y) = x^2+y^2$$

over the domain

$$\mathcal{U} = \{(x,y) \mid x^2+y^2 \leq 1\}.$$

Parametrization: $g : \mathcal{U} \to \mathbb{R}^3$; $g(u,v) = (u, v, u^2+v^2)$

$$\left.\begin{array}{l} \dfrac{\partial g}{\partial u} = (1, 0, 2u) \\ \dfrac{\partial g}{\partial v} = (0, 1, 2v) \end{array}\right\} \Rightarrow \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \underbrace{(-2u, -2v, 1)}_{\text{points "upward"}}$$

$$(F \circ g)(u,v) = (v, -u, (u^2+v^2)^2)$$

$$(F \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) = (u^2+v^2)^2$$

Therefore, we have

$$\int_S F \cdot \vec{n} \, d\sigma := \iint_{\mathcal{U}} (F \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) dA$$

$$= \iint_{\mathcal{U}} (u^2+v^2)^2 \, dA$$

$$= \int_0^{2\pi} \int_0^1 r^4 \cdot r \, dr \, d\theta = \frac{\pi}{3}$$

$\blacksquare$